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COMMENT

Small oscillations of a liquid drop with surface charge

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Abstract. A viscous incompressible non-conducting fluid forms a spherical drop with an electric charge distributed uniformly over its surface. Small oscillations about the spherical shape are possible because of surface tension; a transcendental equation is given, to determine their frequency and damping. If the electric charge is sufficiently large, the drop is unstable. But the principle of exchange of stabilities is valid, and the maximum charge consistent with stability is given by Rayleigh's calculation for a conducting, inviscid fluid drop.

Suppose a viscous incompressible fluid forms a drop in vacuum. It is maintained in a spherical shape by the gravitational attraction of its parts or by surface tension. Small oscillations of the drop about the spherical shape are damped by the viscosity. Their damping and frequency are obtainable from a linearised theory; they are proportional to the real and imaginary parts of a certain dimensionless quantity, here called z^2. The cases of gravity and surface tension were treated by Chandrasekhar (1959) and Reid (1960), respectively. All relevant properties of the drop are characterised by α^4, a dimensionless real parameter which depends on the kind of distortion from the spherical shape. Chandrasekhar (1959) derived a transcendental equation giving z^2 as a multivalued function of α^4, and Reid (1960) showed that the same equation is applicable in his case, with a different formula for α^4. Tang and Wong (1974) have emphasised the wide applicability of Chandrasekhar's transcendental equation. They show that it can be used even when surface tension, gravitational attraction, and electrostatic repulsion are present simultaneously, α^4 being a sum of terms representing the three forces. This result depends on simplifying assumptions about the distribution of electric charge. Tang and Wong (1974) list three cases in which their simple result is obtainable. If the electric charge is distributed uniformly over the surface of the undistorted spherical drop, they obtain Chandrasekhar's equation in the limits of zero and infinite conductivity. However, their treatment of the drop of zero conductivity is based on two questionable assumptions. They ignore the components of electric force parallel to the surface of the drop, and they assert that the electric charge per unit solid angle is constant during the oscillation. If these assumptions are removed, then z^2 depends also on β, a second dimensionless parameter. The modified transcendental equation is given below; it disagrees with that of Saville (1974) because of differences in notation and minor errors in calculation. Since instability of the charged drop is possible, it seems desirable to show that the principle of exchange of stabilities (Chandrasekhar 1961) is applicable. The proof is given below. Using this result, the criterion for stability or instability of the drop can be found. In the absence of gravitation, the criterion is the same as that found by Rayleigh (1882) for a perfectly conducting drop.

Gravitational forces and electrical conductivity are set equal to zero in the calculations of this note, for the sake of brevity and simplicity. Spherical polar coordinates (r, θ, ϕ) are used. The hydrodynamic part of the calculation follows the work of Chandrasekhar (1959), in which the surface of the drop is given by

$$r = R + \epsilon Y_l^m(\theta, \phi). \tag{1}$$

Here ϵ is a time-dependent small parameter, and $Y_l^m(\theta, \phi)$ is a spherical harmonic. Spherically symmetric oscillations are impossible; hence l is a positive integer. Translations of the whole drop are not considered; hence $l > 1$. The fluid velocity is a poloidal vector, whose radial component is

$$u_r = l(l+1)(U(r)/r^2)Y_l^m(\theta, \phi)\exp(-\sigma t),$$

where t is the time. The function $U(r)$ and the complex coefficient σ are to be determined. This radial velocity must be consistent with (1); hence ϵ is proportional to $(U(R)/\sigma)\exp(-\sigma t)$. Since σ will depend on ρ and μ, the density and viscosity of the fluid, the dimensionless quantity

$$z^2 = R^2\rho\sigma/\mu$$

is used. In the calculation of Reid (1960), electric and gravitational forces are absent, and the surface tension T is present; thus z^2 is a multivalued function of l and the dimensionless combination $\rho RT/\mu^2$. If an electric charge Q is distributed over the surface of the drop, there are additional forces proportional to Q^2. The drop of infinite conductivity is treated correctly by Tang and Wong (1974). In this case, z^2 is a function of l and

$$\alpha^4 = l(l-1)(\rho R/\mu^2)[(l+2)T - Q^2/16\pi^2\epsilon_0 R^3], \tag{2}$$

where ϵ_0 is the permittivity of the vacuum. To obtain this result, the surface charge density Σ is calculated from the condition that the electric field vanishes everywhere inside the drop. In the case of zero conductivity, the electric charge is transported only by convection, and $\Sigma - Q/4\pi R^2$ is a small quantity, linear in $U(r)$. These assumptions lead to

$$\partial\Sigma/\partial t = (Q/4\pi R^2)[\partial u_r/\partial r],$$

where the quantity in brackets is evaluated at the surface of the drop. Hence the electrostatic potential outside the drop is

$$\frac{Q}{4\pi\epsilon_0 r} - \frac{l(l+1)QR^{l-3}(R\epsilon_0 U'(R) + l\epsilon_1 U(R))}{4\pi\epsilon_0[l\epsilon_1 + (l+1)\epsilon_0]\sigma r^{l+1}} Y_l^m(\theta, \phi)\exp(-\sigma t),$$

where ϵ_1 is the permittivity of the liquid drop. The electric field inside and outside the drop can be calculated in terms of $U(R)$ and $U'(R)$. The tangential electric force at the surface depends on a non-negative dimensionless parameter

$$\beta = \frac{1}{4\pi\mu R}\left(\frac{l(l+1)\rho Q^2}{l\epsilon_1 + (l+1)\epsilon_0}\right)^{1/2}.$$

The tangential force per unit area, in the direction of increasing θ, is

$$(\beta^2\mu/z^2R^2)[D_1U(r)]_R\, \partial Y_l^m/\partial\theta\, \exp(-\sigma t).$$

Here the subscript R denotes evaluation at $r = R$, and

$$D_1 = d/dr - (l+1)/r$$

is a differential operator. As $\epsilon_1 \to +\infty$, the electric field inside the drop tends to zero, and $\beta \to 0$; hence the case of infinite conductivity can be recovered from the following calculation by setting $\beta = 0$. The pressure inside the drop is a constant plus

$$d/dr(\mu D_2 U(r) + \rho\sigma U(r)) Y_l^m(\theta, \phi) \exp(-\sigma t), \tag{3}$$

where

$$D_2 = d^2/dr^2 - l(l+1)/r^2.$$

The fluid motion inside the drop obeys the linearised Navier–Stokes equation, which leads to (3) and the ordinary differential equation

$$D_2[D_2 U(r) + (\rho\sigma/\mu) U(r)] = 0. \tag{4}$$

The desired solution must satisfy certain boundary conditions. First, $U(r)$ and $U'(r)$ vanish as $r \to 0$. Second, the tangential electric force at the surface requires

$$Rz^2[D_C U(r)]_R - \beta^2 [D_1 U(r)]_R = 0. \tag{5}$$

Here

$$D_C = d^2/dr^2 - 2r^{-1}\, d/dr + l(l+1)/r^2$$

is the differential operator used by Chandrasekhar (1959). Finally, the radial forces at the surface require

$$\frac{(zR)^2}{(l+1)}\left[\frac{d}{dr} D_2 U(r)\right]_R + \frac{z^4 U'(R)}{(l+1)} - 2lz^2\left[\frac{dU}{dr} - \frac{2}{r} U(r)\right]_R$$
$$+ (\alpha^4/R) U(R) - \beta^2 [D_1 U(r)]_R = 0, \tag{6}$$

where α^4 is defined by (2). The conditions (5) and (6) may be compared with the interfacial conditions given by Miller and Scriven (1968), who study a fluid–fluid interface having rather general mechanical properties. But the β^2 terms differ from each of the things treated by Miller and Scriven (1968).

The desired solution of (4) has the form

$$U(r) = Ar^{1/2} J_{l+\frac{1}{2}}(zr/R) + Br^{l+1}.$$

The conditions (5) and (6) give two homogeneous linear equations in A and B. A non-trivial solution is obtained if and only if the determinant vanishes. This condition gives the equation for z:

$$\alpha^4 = -z^4 + 2(l-1)(2l+1)z^2 + \frac{4(l-1)^2(l+1)z^4 Q_{l+\frac{1}{2}}(z)}{(2z^2+\beta^2) Q_{l+\frac{1}{2}}(z) - z^3}, \tag{7}$$

where $Q_{l+\frac{1}{2}}(z) = J_{l+\frac{3}{2}}(z)/J_{l+\frac{1}{2}}(z)$. The right-hand side of (7) is an even meromorphic function of z, and it vanishes as $z \to 0$. If $\beta = 0$, (7) becomes Chandrasekhar's transcendental equation, whose solutions are obtained numerically by Tang and Wong (1974). If β^2 and α^2 are positive, two limiting cases may be studied analytically. When β is small and positive, there is a slowly decaying mode having

$$z^2 = (2l+1)^{-1}\beta^2 + O(\beta^4).$$

The decay time is of the order of the electroviscous time constant used by Lang *et al* (1976); but surface tension is essential here and negligible in their work. When β is large and positive, there are damped oscillations having

$$z^2=(\exp\pm\tfrac{1}{3}\mathrm{i}\pi)\beta^{4/3}+\mathrm{O}(\beta^{2/3}).$$

This implies that vorticity is negligible in the interior of the drop. These oscillations are mentioned by Saville (1974).

A sufficiently large charge can cause the distortion of the spherical surface to increase indefinitely. In the linearised theory, the condition for stability with respect to the distortion described by (1) is $\alpha^4\geqslant 0$, while $\alpha^4<0$ gives instability. The proof of this assertion depends heavily on the principle of exchange of stabilities. Melcher and Schwarz (1968) study a related problem for which this principle is not valid, and it seems desirable to verify this principle. The method is suggested by Chandrasekhar (1961). I assume that σ and z^2 are purely imaginary and non-zero, and proceed to show that any function $U(r)$ which satisfies (4) and the boundary conditions must vanish identically. If (4) is satisfied,

$$\int_0^R \mathrm{d}r(U(r))^*\mathrm{D}_2[\mathrm{D}_2U(r)+(z^2/R^2)U(r)]=0.$$

Partial integration and use of (6) give

$$\begin{aligned}
-\int_0^R \left(\frac{\mathrm{d}U}{\mathrm{d}r}\right)^*\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}^2U}{\mathrm{d}r^2}+\frac{l(l+1)}{r^2}U(r)\right)\right]\mathrm{d}r+\int_0^R\left(\frac{l(l+1)}{r^2}U(r)\right)^*\left(\frac{\mathrm{d}^2U}{\mathrm{d}r^2}\right)\mathrm{d}r\\
+2l(l+1)\int_0^R\left(\frac{\mathrm{d}U}{\mathrm{d}r}\right)^*\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{U(r)}{r^2}\right)\right]\mathrm{d}r\\
+2l(l+1)\int_0^R\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{U(r)}{r^2}\right)^*\right]\left(\frac{\mathrm{d}U}{\mathrm{d}r}\right)\mathrm{d}r+\int_0^R\left|\frac{l(l+1)}{r^2}U(r)\right|^2\mathrm{d}r\\
-(z^2/R^2)\int_0^R[|U'(r)|^2+l(l+1)r^{-2}|U(r)|^2]\,\mathrm{d}r\\
-4l(l+1)|U(R)|^2/R^3-(l+1)(\alpha^4/z^2R^3)|U(R)|^2\\
+(l+1)(\beta^2/z^2R^2)[U(R)]^*[\mathrm{D}_1U(r)]_R=0.
\end{aligned}$$

Further partial integration and use of (5) give

$$\begin{aligned}
\int_0^R|\mathrm{D_C}U(r)|^2\,\mathrm{d}r+(l-1)(l+2)\int_0^R|U'(r)/r|^2\,\mathrm{d}r\\
+3l(l+1)\int_0^R r^{-2}|U'(r)-2U(r)/r|^2\,\mathrm{d}r\\
-(z^2/R^2)\int_0^R[|U'(r)|^2+l(l+1)r^{-2}|U(r)|^2]\,\mathrm{d}r\\
-(l+1)(\alpha^4/z^2R^3)|U(R)|^2-(\beta^2/z^2R)|[\mathrm{D}_1U(r)]_R|^2=0.
\end{aligned}$$

The real and imaginary parts of this expression must vanish separately, and hence the first three integrands must vanish. Therefore $U(r)$ vanishes identically. This means that (7) cannot be satisfied when z^2 is purely imaginary and non-zero. Further calculation shows that the condition for stability of the drop is that $\alpha^4 \geqslant 0$ when $l > 1$. Finally, the necessary and sufficient condition for stability is

$$64\pi^2 T \geqslant Q^2/\epsilon_0 R^3. \tag{8}$$

This agrees with the result of Rayleigh (1882), although Rayleigh considered a fluid of infinite conductivity and zero viscosity.

The foregoing calculations on oscillations and stability should be compared with experiment, if the assumed spherical symmetry of the liquid drop and its electric charge can be realised. It seems desirable to include an arbitrary electrical conductivity in the calculations, and to compare such calculations with experiments on liquids of various conductivities. Such calculations are given by Saville (1974), who seems to doubt whether the principle of exchange of stabilities is applicable. But the calculation of the previous paragraph can easily be generalised to include an arbitrary electrical conductivity, and thus the principle of exchange of stabilities can be proved. Finally, the principle can be used to show that (8) is the condition for stability of a liquid drop of arbitrary conductivity.

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